

Geometric entropy and hagedorn/deconfinement transition

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ABSTRACT: It has recently been proposed that the entanglement entropy can be an order parameter of confinement/deconfinement transitions. To find a clear evidence, we introduce a new quantity called the geometric entropy, which is related to the entanglement entropy via a double Wick rotation. We analyze the geometric entropy and manifestly show that its value becomes discontinuous at the Hagedorn temperature both in the free $\mathcal{N} = 4$ super Yang-Mills and in its supergravity dual.

KEYWORDS: Confinement, AdS-CFT Correspondence, Black Holes.

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1. Introduction

The entanglement entropy has been playing very important roles in recent studies of quantum field theories motivated by both string theory and condensed matter theory. The entanglement entropy S_A can be regarded as a measure of degree of freedom confined in a certain space-like region A , chosen arbitrarily. In the two dimensional conformal field theories, it is indeed proportional to the central charge [1, 2]. In quantum field theory with UV fixed points, we can in general show that the leading ultraviolet divergent term of S_A is proportional to the area of the boundary of A [3], while the subleading terms depend on the shape of the region A . Even though the direct computation of S_A often involves complicated analysis, the holographic calculation [4] based on AdS/CFT correspondence [5] provides a more tractable way of doing this (for recent progresses see e.g. [6]–[41]).

In the analysis of quantum phase transitions which frequently appear in condensed matter systems, it has been pointed out that the entanglement entropy can be used as a quantum order parameter [42–44]. Especially it is quite useful to specify the phases in topological theories as ordinary correlation functions become trivial [43, 44], while the entanglement entropy (called topological entanglement entropy¹) does not.

¹Refer to [40] for a holographic calculation of topological entanglement entropy.

In the recent papers [14, 23], the entanglement entropy in confining gauge theories has been studied holographically and it has been shown that it undergoes a sort of phase transition when we change the size of A . This behavior has been confirmed recently in the lattice gauge theories [32, 37]. Thus it is natural to expect that the entanglement entropy can be an order parameter of the confinement/deconfinement transition in gauge theories [14, 23]. In order to reinforce this idea, the main purpose of this paper is to show that the entanglement entropy (or more generally, the geometric entropy) is a nice order parameter of the confinement/deconfinement transition in $\mathcal{N} = 4$ super Yang-Mills theory when we change the temperature. This phase transition is well-known to be dual to the Hagedorn transition in string theory via the AdS/CFT.

Especially we will employ the free Yang-Mills analysis in [45] and compute a certain entropy defined later, which is closely related to the entanglement entropy. This quantity is not exactly the same as the ordinary entanglement entropy, but can be regarded as its double Wick rotated one. Since it is defined geometrically, we will call this quantity the geometric entropy in this paper.² The relation between our geometric entropy and the entanglement entropy is analogous to the one between the Wilson loop and the Polyakov loop. Remember that the Polyakov loop is strictly speaking not well-defined on a compact manifold because it introduces a positive charge, while our geometric entropy is well-defined. It has also a nice property that its gravity dual is easy to analyze. We explicitly examine its behavior and show that its value jumps at the transition point. This behavior qualitatively agrees with the results in the supergravity dual. We will also discuss that this quantity can be a useful order parameter in other theories such as the topological field theories and two dimensional Yang-Mills theory.

This paper is organized as follows: in section two, we define the geometric entropy as a certain double Wick rotation of the entanglement entropy. We also calculate this quantity holographically in the AdS_5 back hole background. In section three, we compute the geometric entropy in the free $\mathcal{N} = 4$ Yang-Mills theory and compare the results with its dual gravity result. In section four, we briefly discuss the application of the geometric entropy to topological field theories and two dimensional Yang-Mills. In section five, we summarize our conclusion.

2. Geometric Entropy in $\mathcal{N} = 4$ SYM and AdS/CFT

2.1 Definition of geometric entropy

We compactify on S^3 a four dimensional quantum field theory such as the $\mathcal{N} = 4$ super Yang-Mills. At finite temperature $T = \frac{1}{\beta}$, it is defined on $S^1 \times S^3$. We express the metric

²In some literature, the authors defined the geometric entropy to be exactly the same as the entanglement entropy. However, in this paper, we use the term ‘geometric entropy’ in a broader sense.

of S^3 as follows

$$d\Omega_{(3)}^2 = d^2\theta + \sin^2\theta(d\psi^2 + \sin^2\psi d\phi^2), \quad (2.1)$$

where $0 \leq \theta, \psi \leq \pi$ and $0 \leq \phi \leq 2\pi$.

Now we change the periodicity of ϕ into $0 \leq \phi \leq 2\pi k$. For $k \neq 1$, there exist conical singularities at $\psi = 0, \pi$ with the deficit angle $\delta = 2\pi(1-k)$. The submanifold of S^3 defined by these singular points is equal to S^1 (the largest circle of S^3).

The partition function on this singular space is defined to be $Z_{\text{YM}}(k)$. The ordinary partition function on $S^1 \times S^3$ coincides with $Z_{\text{YM}}(1)$. We can consider the normalized partition function and can regard it as follows

$$\frac{Z_{\text{YM}}(k)}{(Z_{\text{YM}}(1))^k} = \text{Tr}\rho^k, \quad (2.2)$$

where $\rho = e^{-2\pi H}$ is the density matrix when we regard the coordinate ϕ as the Euclidean time (and H is its Hamiltonian) via the double Wick rotation.

Then, following the definition of von-Neumann entropy, we define the geometric entropy S_G by

$$S_G = -\text{Tr}\rho \log \rho = -\frac{\partial}{\partial k} \log \left[\frac{Z_{\text{YM}}(k)}{(Z_{\text{YM}}(1))^k} \right] \Bigg|_{k=1}. \quad (2.3)$$

As is clear from the above, the geometric entropy is different from the ordinary thermal entropy but is naively related to it via the double Wick rotation. In other words, in the ordinary entropy we regard the thermal circle S^1 is the Euclidean time, while in our geometric entropy we regarded ϕ in S^3 as the Euclidean time. Thus it is analogous to the Polyakov loop instead of the Wilson loop. Remember that the Polyakov loop is strictly speaking not well-defined on a compact manifold, while our geometric entropy is well-defined.

However, strictly speaking, we should regard S_G as a double Wick rotated entanglement entropy rather than a double Wick rotated thermal entropy. This is because we actually have fixed points under the periodic identification of ϕ , which leads to the deficit angle circle. If we fix the value of θ , then the geometry (2.1) is equivalent to a two sphere with a cut which extends from the North pole $\psi = 0$ to South pole $\psi = \pi$. If we regard one of the coordinates of the two sphere as the Euclidean time, this is exactly what we typically encounter when we compute the entanglement entropy in quantum field theories (see [2]). In this way we can say that the geometric entropy is locally equivalent to the entanglement entropy.³ It will also be an interesting future problem to compute the ordinary entanglement entropy itself to see if it can be an order parameter, though this will require a more complicated analysis.

³We may equally say that the geometric entropy is the entanglement entropy when we consider the quantum field theory on the double Wick rotated spacetime. Since the metric of this spacetime is rather complicated we will not present its detail in this paper.

In the practical computations of S_G , it is convenient to take the values of k to be fractional $k = \frac{1}{n}$. This theory is equivalent to the $\mathcal{N} = 4$ super Yang-Mills on the orbifold S^3/Z_n . The Z_n identification is simply defined by $\phi \sim \phi + \frac{2\pi}{n}$ in the coordinate (2.1). This is locally the same as the \mathbb{C}/Z_n orbifold and thus is non-supersymmetric. The careful analysis of the spin structure [46] shows that n should be an odd integer. Then we can calculate the geometric entropy from the formula

$$S_G = -\frac{\partial}{\partial(1/n)} \log \left[\frac{Z_{\text{YM}}(S^3/Z_n)}{(Z_{\text{YM}}(S^3))^{1/n}} \right] \Bigg|_{n=1}. \quad (2.4)$$

The partition function in the $\mathcal{N} = 4$ Yang-Mills theory at finite temperature and coupling has not been obtained so far. Therefore, in the next section, we will perform the calculation of the partition function and find the geometric entropy in the free $\mathcal{N} = 4$ Yang-Mills theory. As we will see later, even under this free field approximation, we can still reproduce qualitative behavior of S_G expected from the gravity computation.

2.2 Holographic Calculation of S_G

We would like to first compute S_G in the dual gravity side. In the Yang-Mills language, the supergravity analysis is dual to the strongly coupled Yang-Mills.

If we require that the boundary is given by $S^1 \times S^3$, then only two examples are known as the bulk space [47]; one is the thermal AdS

$$ds^2 = \left(\frac{r^2}{R^2} + 1 \right) d\tau^2 + \frac{dr^2}{\frac{r^2}{R^2} + 1} + r^2 d\Omega_{(3)}^2, \quad (2.5)$$

and the other one is the AdS (large) black hole

$$ds^2 = \left(\frac{r^2}{R^2} + 1 - \frac{M}{r^2} \right) d\tau^2 + \frac{dr^2}{\frac{r^2}{R^2} + 1 - \frac{M}{r^2}} + r^2 d\Omega_{(3)}^2. \quad (2.6)$$

The horizon of the latter spacetime is at r_+ defined by $\frac{r_+^2}{R^2} + 1 - \frac{M}{r_+^2} = 0$.

By requiring the smoothness of the Euclidean geometry, we find that the periodicity of τ (2.6) is given by

$$\tilde{\beta} = \frac{2\pi r_+ R^2}{2r_+^2 + R^2}. \quad (2.7)$$

The analysis of the free energy shows that at low temperature $\tilde{\beta} > \tilde{\beta}_H$ the thermal AdS solution is stable, while at high temperature $\tilde{\beta} < \tilde{\beta}_H$ the AdS black hole solution becomes favored [47]. Here the phase transition temperature is given by $\beta_H = \frac{2\pi}{3}R$ and is known as the Hawking-Page transition.

Now we would like to compute S_G . In order for this we need to put the deficit angle $2\pi(1 - \alpha)$ along the circle S^1 on S^3 . The presence of the codimension two deficit angle

leads to the delta functional source of the scalar curvature $R = 4\pi(1 - \alpha)\delta(x)$. If we plug this into the Einstein-Hilbert action, we get

$$S_{\text{sugra}} = -\frac{1}{16\pi G_N^{(5)}} \int \sqrt{g} R + \dots = -\frac{\text{Area}(\gamma)}{4G_N^{(5)}}(1 - \alpha), \quad (2.8)$$

where γ is the codimension two surface where the deficit angle is localized (these arguments are very similar to the one in [8]). Using the bulk to boundary relation $Z_{\text{CFT}} = Z_{\text{sugra}} = e^{-S_{\text{sugra}}}$ in the supergravity approximation [48, 49], we eventually be able to obtain the geometric entropy as follows

$$S_G = -\frac{\partial}{\partial \alpha} \log \frac{Z_{\text{sugra}}(\alpha)}{(Z_{\text{sugra}}(0))^\alpha} = \frac{\partial}{\partial \alpha} S_{\text{sugra}} = \frac{\text{Area}(\gamma)}{4G_N^{(5)}}. \quad (2.9)$$

The surface γ for the geometric entropy S_G is given by the codimension three surface defined by $\sin \psi = 0$, which extends in the (τ, r, θ) direction. We put the UV cut off at $r = r_\infty \gg R$.

In the thermal AdS, we find

$$S_G^{\text{ads}} = \frac{1}{4G_N^{(5)}} \int_0^{\tilde{\beta}} \int_0^{r_\infty} r dr \int_0^{2\pi} d\theta = \frac{\pi \tilde{\beta} r_\infty^2}{4G_N^{(5)}}, \quad (2.10)$$

while in the AdS black hole we get

$$S_G^{\text{bh}} = \frac{1}{4G_N^{(5)}} \int_0^{\tilde{\beta}} \int_{r_+}^{r_\infty} r dr \int_0^{2\pi} d\theta = \frac{\pi \tilde{\beta} (r_\infty^2 - r_+^2)}{4G_N^{(5)}}. \quad (2.11)$$

For the large AdS black hole we have the relation equivalent to (2.7)

$$r_+ = \frac{\pi R^2}{2\tilde{\beta}} + \sqrt{\frac{\pi^2 R^4}{4\tilde{\beta}^2} - \frac{R^2}{2}}. \quad (2.12)$$

Notice that the AdS BH exists when $\tilde{\beta} < \frac{\pi R}{\sqrt{2}}$. Below we introduce the dimensionless temperature β defined by

$$\beta = \frac{\tilde{\beta}}{R}. \quad (2.13)$$

We are interested in the difference⁴ of these entropies. This quantity is vanishing at the temperature lower than the Hagedorn transition, i.e. $\beta > \frac{2\pi}{3}$. On the other hand, at high temperature ($\beta < \frac{2\pi}{3}$), we obtain the non-vanishing result

$$\Delta S_G = -\frac{\pi \tilde{\beta} r_+^2}{4G_N^{(5)}} = -\frac{\pi R^3}{4G_N^{(5)}} \cdot \frac{\left(\pi + \sqrt{\pi^2 - 2\beta^2}\right)^2}{16\beta}. \quad (2.14)$$

⁴In the original arguments of the Hawking-Page transition, we needed to choose slightly different temperature between thermal AdS and AdS BH. This subtlety is not important in our argument.

If we assume the $AdS_5 \times S^5$ background in type IIB string dual to the $\mathcal{N} = 4$ super Yang-Mills, we can rewrite the expression as follows⁵ (using $\frac{R^3}{G_N^{(5)}} = \frac{2N^2}{\pi}$)

$$\begin{aligned} \Delta S_G &= -\frac{\pi^2 N^2}{8\beta} \left(1 + \sqrt{1 - \frac{2\beta^2}{\pi^2}} \right)^2 \\ &\simeq -\frac{N^2}{2} \left(\frac{\pi^2}{\beta} - \beta \right) + O(\beta^3). \end{aligned} \tag{2.15}$$

If we start with low temperature and increase the temperature gradually, then at $\beta = \frac{2\pi}{3}$, the quantity ΔS_G suddenly jumps from zero to $-\frac{2\pi^2 N^2}{9\beta}$ (see the figure 1(b) in the next section). Thus we can conclude that S_G is an order parameter of the confinement/deconfinement transition. Our analysis can be comparable to that of the holographic entanglement entropy in a confining gauge theories at zero temperature in [14, 23, 40]. In the latter case, the derivative of the entropy (not the entropy itself) jumps when we change the size of the subsystem A which defines the entanglement entropy A .

3. Geometric entropy in free $\mathcal{N} = 4$ super Yang-Mills

We would like to return to the Yang-Mills analysis. Since the evaluation of the partition function in the interacting $\mathcal{N} = 4$ super Yang-Mills is rather difficult, here we would like to be satisfied with the free Yang-Mills calculation. As noticed in [45], the free Yang-Mills analysis can capture the confinement/deconfinement transition since the Gauss law constraint on S^3 restricts the total charge to be vanishing.

The partition function in the free Yang-Mills theory is written in the form [45] (we set $x = e^{-\beta}$)

$$Z_{\text{YM}} = \int [dU] e^{\sum_{m=1}^{\infty} \frac{1}{m} (z_s(x^m) + z_v(x^m) + (-1)^{m+1} z_f(x^m)) \text{tr}(U^m) \text{tr}((U^\dagger)^m)}, \tag{3.1}$$

$$= \int \prod_{i=1}^N d\theta_i e^{-\sum_{i \neq j} V(\theta_i - \theta_j)}, \tag{3.2}$$

where we diagonalized the unitary matrix U in the final expression. Also $z_s(x)$, $z_v(x)$ and $z_f(x)$ denote the single particle partition functions of scalars, vectors and fermions in a given gauge theory.

The potential $V(\theta)$ is given as follows

$$V(\theta) = \log 2 + \sum_{m=1}^{\infty} V_m \cos(m\theta), \tag{3.3}$$

⁵In the final expression we performed the high temperature expansion. Notice that the leading term $\sim \beta^{-1}$ agrees with the result $\Delta S_A = -\frac{\pi N^2 L}{4V_1}$ in [14] by identifying $\beta = 2\pi \frac{L}{V_1}$, which is obtained by looking at conformally invariant quantities. This agreement is because the high temperature limit $\beta \rightarrow 0$ means that the size of the sphere S^3 becomes infinitely large and so we can regard it as R^3 .

where we set

$$V_m = \frac{1}{m}(1 - z_s(x^m) - z_v(x^m) - (-1)^{m+1}z_f(x^m)). \quad (3.4)$$

In the $\mathcal{N} = 4$ SYM on the orbifold S^3/Z_n , assuming n is an odd integer, we find

$$z_s(x) = 6 \frac{x(1+x^n)}{(1-x)^2(1-x^n)}, \quad z_v(x) = \frac{2x^2(1+2x^{n-1}-x^n)}{(1-x)^2(1-x^n)}, \quad z_f(x) = \frac{16x^{\frac{n}{2}+1}}{(1-x)^2(1-x^n)}. \quad (3.5)$$

For the detailed derivation of these functions, please refer to the appendix A. A calculation of $z(x)$ in a different orbifold has been done in [50].

To solve the matrix model, we introduce the density of the eigenvalues as usual

$$\rho(\theta) = \frac{1}{2\pi} + \sum_{m=1}^{\infty} \frac{\rho_m}{\pi} \cos(m\theta), \quad (3.6)$$

so that it is normalized as $\int_{-\pi}^{\pi} d\theta \rho(\theta) = 1$. Then the free energy looks like

$$\beta F = \beta E_0 + N^2 \sum_{m=1}^{\infty} |\rho_m|^2 V_m, \quad (3.7)$$

where E_0 is the Casimir energy. The contribution from the Casimir energy is not important for our purpose. This is because we always subtract the result from the one with the periodic boundary condition in the Euclidean time direction as we did so in the gravity side and therefore the Casimir energy part cancels out.

At low temperature (i.e. confining phase), we find $\rho_{n \geq 1} = 0$. When $T = T_H$ (the Hagedorn transition point, i.e. $V_1(x) = 0$), ρ_1 jumps to $\rho_1 = 1$. On the other hand, in the high temperature limit, the eigenvalue distribution becomes delta-functional $\rho(\theta) = \delta(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\theta}$ and thus $\rho_{n \geq 1} = 1$.

3.1 Analysis near the transition

In the low temperature case, we can assume only ρ_1 becomes non-zero. The density of the eigenvalues can be solved as [45]

$$\rho(\theta) = \frac{1}{\pi \sin^2\left(\frac{\theta_0}{2}\right)} \sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)} \cos \frac{\theta}{2}, \quad (3.8)$$

$$\sin^2\left(\frac{\theta_0}{2}\right) = 1 - \sqrt{1 - \frac{1}{z_s(x) + z_v(x) + z_f(x)}}, \quad (3.9)$$

where $\rho(\theta)$ has the support on $\{-\theta_0 < \theta < \theta_0\}$.

Putting (3.8) into (3.7), we find the free energy in fairly simple form

$$\beta F = -N^2 \left(\frac{1}{2 \sin^2\left(\frac{\theta_0}{2}\right)} + \frac{1}{2} \log \left(\sin^2\left(\frac{\theta_0}{2}\right) \right) - \frac{1}{2} \right). \quad (3.10)$$

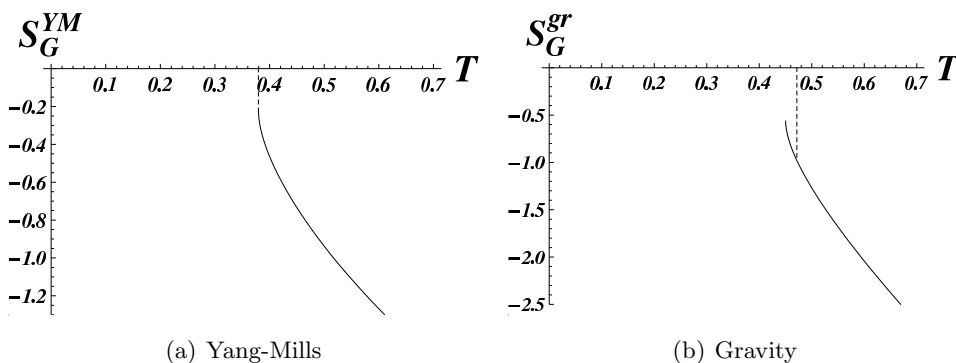


Figure 1: The behavior of $\Delta S_G/N^2$ for free Yang-Mills (a) and IIB supergravity (b). The horizontal axis corresponds to the temperature T . The temperature at which the phase transition occurs is $T_H = -1/\ln(7 - 4\sqrt{3}) = 0.379$ in the Yang-Mills theory and $T_H = 3/2\pi = 0.477$ (dashed line) in the dual gravity. Notice that the line starts at $T = \sqrt{2}/\pi = 0.450$ in (b) above which the AdS black hole exists.

For $T > T_H$, this action is well-defined (see (3.9)) and takes values of order N^2 , while this is order one for $T < T_H$ because the coefficients V_n appeared in (3.7) are positive and the minimal configuration $\rho_{n \geq 1} = 0$ gives $F = 0$.

The geometric entropy can be computed from the partition function Z in the free Yang-Mills by the following formula as defined in (2.4)

$$\begin{aligned} \Delta S_G &= -\frac{\partial}{\partial(1/n)} \left(\log Z(n) - \frac{1}{n} \log Z(1) \right) \Big|_{n=1} \\ &= -\frac{\partial}{\partial n} \left((\beta F)(n) - \frac{1}{n} (\beta F)(1) \right) \Big|_{n=1}. \end{aligned} \tag{3.11}$$

Using the single particle partition function (3.5), we can easily plot the geometric entropy near the phase transition as in figure 1(a), where we also plot the gravity result (2.15) in figure 1(b). They look qualitatively similar and show a jump at the transition point. One may notice that the value of $\frac{dS_G}{dT}$ is infinite in the free $\mathcal{N} = 4$ Yang-Mills, while it is finite in the gravity. This difference comes from the following fact. In gravity side, the temperature at which the black hole solution appears and the temperature at which the black hole becomes stable against the thermal AdS, are different. However, in the free Yang-Mills limit, they do degenerate as is clear from the fact that there is no saddle point or local minima in the free energy (3.7).

It is also straightforward to take the chemical potential (μ_1, μ_2, μ_3) of the R -charges (Q_1, Q_2, Q_3) into account by multiplying $z_s(x)$ and $z_f(x)$ by the factor $(e^{\mu_1} + e^{-\mu_1} + e^{\mu_2} + e^{-\mu_2} + e^{\mu_3} + e^{-\mu_3})/6$ and $(e^{\frac{\mu_1}{2}} + e^{-\frac{\mu_1}{2}})(e^{\frac{\mu_2}{2}} + e^{-\frac{\mu_2}{2}})(e^{\frac{\mu_3}{2}} + e^{-\frac{\mu_3}{2}})/8$, respectively. The matrix model description and its phase structure in the R -charged case have been worked out in [52–54]. We focus on the specific case $(\mu_1, \mu_2, \mu_3) = (\mu, 0, 0)$, and the result is plotted in figure 2. Even though the transition temperature decreases as μ becomes large

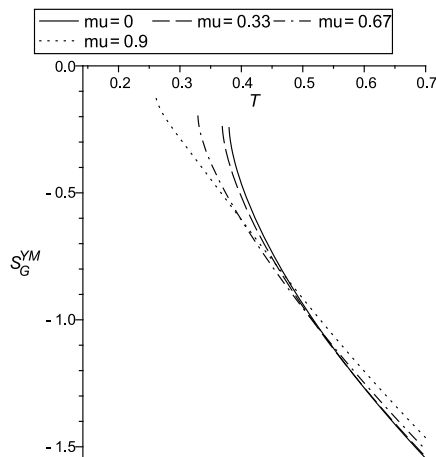


Figure 2: The behavior of $\Delta S_G/N^2$ for free Yang-Mills with the specified values of the chemical potential $(\mu_1, \mu_2, \mu_3) = (\mu, 0, 0)$ of the R -charges.

($\mu < 1$), the discontinuity of S_G is still present. A more non-trivial extension will be to introduce a potential with respect to the rotation in the S^3 , which is dual to a rotating black hole. The analysis of the phase structure in the rotating system has been recently done [55] (the same system in decoupling limit was studied in [56]).

3.2 Analysis of high temperature limit

In the high temperature limit $x \rightarrow 1$ ($\beta \rightarrow 0$), the $z_s(x^m)$, $z_v(x^m)$ and $z_f(x^m)$ behave like

$$\begin{aligned} z_s(x^m) &\simeq \frac{12}{m^3 n \beta} + \frac{n^2 - 1}{m n \beta} + O(\beta), \\ z_v(x^m) &\simeq \frac{4}{m^3 n \beta^3} + \frac{n^2 - 6n - 1}{3 m n \beta} + O(1), \\ z_f(x^m) &\simeq \frac{16}{m^3 n \beta^3} - \frac{2(2 + n^2)}{3 m n \beta} + O(\beta). \end{aligned} \tag{3.12}$$

Then we can evaluate the free energy as follows

$$\begin{aligned} -\beta F_{\text{scalar}} &\simeq \frac{N^2}{30 n \beta^3} (4\pi^4 - 5\pi^2 \beta^2 + 5n^2 \pi^2 \beta^2), \\ -\beta F_{\text{vector}} &\simeq \frac{N^2}{90 n \beta^3} (4\pi^4 - 5\pi^2 \beta^2 - 30n\pi^2 \beta^2 + 5n^2 \pi^2 \beta^2), \\ -\beta F_{\text{fermion}} &\simeq \frac{N^2}{90 n \beta^3} (14\pi^4 - 10\pi^2 \beta^2 - 5n^2 \pi^2 \beta^2). \end{aligned} \tag{3.13}$$

In the end, the geometric entropy for each fields is found⁶ to be

$$S_G^{\text{scalar}} = \frac{\pi^2 N^2}{3\beta}, \quad S_G^{\text{vector}} = -\frac{2\pi^2 N^2}{9\beta}, \quad S_G^{\text{fermion}} = -\frac{\pi^2 N^2}{9\beta}. \tag{3.15}$$

⁶These can be comparable to the entanglement entropy computed in [14] (setting $\beta = 2\pi \frac{L}{v_1}$)

$$S_{\text{ent}}^{\text{scalar}} = \frac{\pi^2 N^2}{3\beta}, \quad S_{\text{ent}}^{\text{vector}} = \frac{\pi^2 N^2}{9\beta}, \quad S_{\text{ent}}^{\text{fermion}} = -\frac{\pi^2 N^2}{9\beta}. \tag{3.14}$$

Notice also that the total sum is vanishing $S_G^{\mathcal{N}=4SYM} = S_G^{\text{scalar}} + S_G^{\text{vector}} + S_G^{\text{fermion}} = 0$.

In the high temperature limit, we need to subtract S_G^P from the above result, where S_G^P is the geometric entropy in the case where the fermions obey the periodic boundary condition in the S^1 direction so that the partition function becomes $\text{tr}(-1)^F e^{-\beta H}$. This is because in the gravity side calculation, we considered the difference⁷ between the result in the AdS black hole and the one in the thermal AdS.

In the periodic case, we find the total free energy becomes in the high temperature limit

$$-\beta F_{\text{tot}}^P = \frac{\pi^2 N^2 (n-1)}{3\beta}, \tag{3.16}$$

which leads to the entropy

$$S_G^P = \frac{\pi^2 N^2}{3\beta}. \tag{3.17}$$

We would like to claim the difference

$$\Delta S_G = S_G^A - S_G^P = -\frac{\pi^2 N^2}{3\beta}, \tag{3.18}$$

should be comparable to the supergravity result (2.15), which differs with each other by the factor $\frac{2}{3}$. This is analogous to the $\frac{4}{3}$ factor in a similar ratio of the thermal entropy [57] (a similar ratio in $\mathcal{N} = 1$ conformal field theories has been worked out recently in [16]).

In this way, we have shown that the geometric entropy in free $\mathcal{N} = 4$ Yang-Mills qualitatively (or semi-qualitatively) agrees with that in its holographic dual. Our result provides a strong support that the geometric entropy is a nice order parameter of Hagedorn/deconfinement phase transition.

4. Geometric entropy in TQFT and 2D YM

As we have seen, the geometric entropy successfully plays the role of order parameter in the $\mathcal{N} = 4$ Yang-Mills. Another advantage of considering this quantity is that we can define the geometric entropy in any Euclidean field theory, even if the spacetime is not a direct product of the (Euclidean) time times a space manifold. A typical such example is the quantum field theory on S^2 . The geometric entropy is defined by introducing the deficit angle at two points on the sphere, e.g. the North and South Pole, in a similar way we did for S^3 .

If we consider a two dimensional topological field theory defined on a Riemann surface Σ_g , then the partition function Z_g depends only on the genus g and not on the other

Notice that these results agree with each other except the gauge field. This will be due to the subtle issue raised in the paper by [51]. If we literally evaluate the entanglement entropy from the partition function, we find the result in (3.15) for the vector field. However, if we eliminate a surface term we get the result in (3.14) which is the twice of the real scalar field result.

⁷This procedure is not necessary in the analysis of the low temperature region since it become a minor contribution.

geometrical parameter or moduli. We can define the geometric entropy by introducing a cut on Σ_g . This leads to the n -sheeted Riemann surface. Then this n -sheeted surface has the genus $G = ng$. The position of the cut is not important as the theory is topological. In the end, the entropy is defined as follows

$$S_G(g) = -\frac{\partial}{\partial n} \log \left[\frac{Z_{ng}}{(Z_g)^n} \right] \Bigg|_{n=1}. \quad (4.1)$$

Especially, in the sphere case $g = 0$, we simply find

$$S_G(0) = \log Z_0. \quad (4.2)$$

A similar result can be found for the three dimensional topological field theory such as the Chern-Simons gauge theory on a three sphere. By putting the deficit angle along a circle, the geometric entropy becomes (see [58])

$$S_G(S^3) = \log Z(S^3). \quad (4.3)$$

This is exactly the same as the topological entanglement entropy introduced in [43, 44].

A more interesting example may be the $U(N)$ two dimensional Yang-Mills theory. It has been shown that the system undergoes a third order phase transition [59] by computing the partition function exactly using the well-known formula

$$Z(g, A) = \sum_R (\dim R)^{2-2g} e^{-\frac{g^2}{2N} C_2(R) \tilde{A}}. \quad (4.4)$$

Here \tilde{A} is the area of the Riemann manifold; R is a representation of $U(N)$. Below we measure the area in units of $1/g^2$ i.e. $A = \tilde{A}g^2$.

Now we would like to see if the geometric entropy can be regarded as an order parameter. We concentrate on the genus 0 case and put a cut between the North and South pole. Then the geometric entropy is given by

$$S_G(A) = -\frac{\partial}{\partial n} \log \left[\frac{Z(nA)}{(Z(A))^n} \right] \Bigg|_{n=1} = N^2 (AF'(A) - F(A)), \quad (4.5)$$

where $N^2 F(A) = -\log Z(A)$. By employing the analytic expressions of the free energy $F(A)$ in [59], we can compute the gap between S_G in the strongly coupled phase $A > A_c$ and the weakly coupled on $A < A_c$, where A_c is the value of A where the phase transition occurs. It behaves like

$$\Delta S_G(A) \sim N^2 (A - A_c)^2. \quad (4.6)$$

Therefore, in this example, we can regard $\frac{d^2 S_G(A)}{dA^2}$ as an order parameter of the phase transition in the two dimensional Yang-Mills. In other words, the geometric entropy is an analogue of the thermodynamical entropy for a quantum field theory on a general Euclidean manifolds.

5. Conclusion

In this paper we introduced a new quantity called the geometric entropy in quantum field theories, especially focusing on the gauge theories which often have their holographic duals via AdS/CFT. This quantity is analogous to the Polyakov loop and indeed we defined it by a double Wick rotation of another basic quantity known as the entanglement entropy.

The main claim of this paper is that the geometric entropy can be used as an order parameter of Hagedorn/deconfinement phase transitions. We explicitly examined the geometric entropy in both Yang-Mills theory and its AdS dual and showed that this claim is indeed true. We also noticed that this quantity plays the role of order parameter in the two dimensional Yang-Mills theory. It will be an intriguing future direction to investigate other phase transitions from the viewpoint of the geometric entropy.

The advantage of considering the geometric entropy is that it is a universal physical quantity because we can define this quantity in any quantum field theories even if they are not gauge theories. It gives much more detailed information than the thermal entropy and the energy stress tensor do. Therefore it will be very interesting to understand the holography in more general spacetimes such as the de-Sitter space by using the geometric entropy as a probe.

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A. Computation of the single particle partition functions

Let's choose the metric of S^3 as

$$d\Omega_{(3)}^2 = d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2, \tag{A.1}$$

where $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi, \psi \leq 2\pi$. This can be embedded in \mathbb{C}^2 as

$$z_1 = \sin \theta e^{i\phi}, \quad z_2 = \cos \theta e^{i\psi}. \tag{A.2}$$

When we take the Z_n orbifold of S^3 , $\psi \simeq \psi + \frac{2\pi}{n}$, the \mathbb{C}^2 coordinates become identified⁸ as $z_2 \simeq e^{i\frac{2\pi}{n}} z_2$. The Z_n action acts on the field \mathcal{O} [60]

$$e^{2\pi i \frac{a-b}{n}} \mathcal{O}_{ab} = e^{2\pi i \frac{J_2}{n}} \mathcal{O}_{ab}, \quad (a, b = 1, \dots, n), \tag{A.3}$$

⁸A similar calculations in a different orbifold can be found in [50].

where we denote $SO(4)$ generators as J_1, J_2 which act z_1, z_2 plane respectively. Decomposing $SO(4) \simeq SU(2)_L \times SU(2)_R$ and introducing $m_L \equiv J_1 + J_2$, $m_R \equiv J_1 - J_2$, the Z_n invariant modes satisfy

$$m_L - m_R = a - b \pmod{n}. \quad (\text{A.4})$$

Here we take trivial modes ($a = b$) as invariant states, which satisfy $m_L - m_R = n\mathbb{Z}$. The single particle partition function for the scalar field can be represented as the summation of the invariant state with the representation $(m_L, m_R) = (j, j)$ and the energy $E = 2j + 1$

$$\begin{aligned} z_s(x) &= 6 \sum_{j=0,1/2,\dots} \sum_{m_L=-j}^j \sum_{m_R=-j}^j x^{2j+1} y^{m_L-m_R} |_{m_L-m_R=n\mathbb{Z}, y=1} \\ &= 6 \frac{x(1+x^n)}{(1-x)^2(1-x^n)}. \end{aligned} \quad (\text{A.5})$$

Similar calculation leads the single particle partition function for the vector field

$$\begin{aligned} z_v(x) &= \sum_{j=0,1/2,\dots} \sum_{m_L=-j-1}^{j+1} \sum_{m_R=-j}^j x^{2j+2} y^{m_L-m_R} |_{m_L-m_R=n\mathbb{Z}, y=1} + (m_L \leftrightarrow m_R) \\ &= \frac{2x^2(1+2x^{n-1}-x^n)}{(1-x)^2(1-x^n)}. \end{aligned} \quad (\text{A.6})$$

For fermions, there are two kinds of the Z_n action

$$g = e^{2\pi i \frac{J_2}{n}} \quad \text{or} \quad e^{2\pi i \frac{n+1}{n} J_2}, \quad (\text{A.7})$$

but we must take the latter with $n = 2l + 1$ to require $g^n = 1$. Hence the invariant fermionic states satisfy

$$(2n+1)(m_L - m_R) = 0, \pmod{n}, \quad (\text{A.8})$$

where we take trivial modes as we did in the bosonic computation. Since $m_L - m_R$ takes half-integer for the fermion, (A.8) becomes

$$m_L - m_R = n \left(\mathbb{Z} + \frac{1}{2} \right). \quad (\text{A.9})$$

The resulting single particle partition function becomes

$$\begin{aligned} z_f(x) &= 4 \left\{ \sum_{j=0,1/2,\dots} \sum_{m_L=-j-1/2}^{j+1/2} \sum_{m_R=-j}^j x^{2j+3/2} y^{m_L-m_R} |_{m_L-m_R=n(\mathbb{Z}+\frac{1}{2}), y=1} + (m_L \leftrightarrow m_R) \right\} \\ &= \frac{16x^{1+\frac{n}{2}}}{(1-x)^2(1-x^n)}. \end{aligned} \quad (\text{A.10})$$

References

- [1] C. Holzhey, F. Larsen and F. Wilczek, *Geometric and renormalized entropy in conformal field theory*, *Nucl. Phys. B* **424** (1994) 443 [[hep-th/9403108](#)].
- [2] P. Calabrese and J.L. Cardy, *Entanglement entropy and quantum field theory*, *J. Stat. Mech.* (2004) P06002 [[hep-th/0405152](#)].
- [3] L. Bombelli, R.K. Koul, J.-H. Lee and R.D. Sorkin, *A quantum source of entropy for black holes*, *Phys. Rev. D* **34** (1986) 373;
M. Srednicki, *Entropy and area*, *Phys. Rev. Lett.* **71** (1993) 666 [[hep-th/9303048](#)].
- [4] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, *Phys. Rev. Lett.* **96** (2006) 181602 [[hep-th/0603001](#)]; *Aspects of holographic entanglement entropy*, *JHEP* **08** (2006) 045 [[hep-th/0605073](#)].
- [5] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)].
- [6] R. Emparan, *Black hole entropy as entanglement entropy: a holographic derivation*, *JHEP* **06** (2006) 012 [[hep-th/0603081](#)].
- [7] Y. Iwashita, T. Kobayashi, T. Shiromizu and H. Yoshino, *Holographic entanglement entropy of de Sitter braneworld*, *Phys. Rev. D* **74** (2006) 064027 [[hep-th/0606027](#)].
- [8] D.V. Fursaev, *Proof of the holographic formula for entanglement entropy*, *JHEP* **09** (2006) 018 [[hep-th/0606184](#)].
- [9] S.N. Solodukhin, *Entanglement entropy of black holes and AdS/CFT correspondence*, *Phys. Rev. Lett.* **97** (2006) 201601 [[hep-th/0606205](#)].
- [10] T. Hirata and T. Takayanagi, *AdS/CFT and strong subadditivity of entanglement entropy*, *JHEP* **02** (2007) 042 [[hep-th/0608213](#)];
M. Headrick and T. Takayanagi, *A holographic proof of the strong subadditivity of entanglement entropy*, *Phys. Rev. D* **76** (2007) 106013 [[arXiv:0704.3719](#)].
- [11] P. Fendley, M.P.A. Fisher and C. Nayak, *Topological entanglement entropy from the holographic partition function*, *J. Stat. Phys.* **126** (2007) 1111 [[cond-mat/0609072](#)].
- [12] R. Brustein, M.B. Einhorn and A. Yarom, *Entanglement and nonunitary evolution*, *JHEP* **04** (2007) 086 [[hep-th/0609075](#)].
- [13] H. Casini, *Mutual information challenges entropy bounds*, *Class. and Quant. Grav.* **24** (2007) 1293 [[gr-qc/0609126](#)].
- [14] T. Nishioka and T. Takayanagi, *AdS bubbles, entropy and closed string tachyons*, *JHEP* **01** (2007) 090 [[hep-th/0611035](#)].
- [15] J.-W. Lee, J. Lee and H.-C. Kim, *Dark energy from vacuum entanglement*, *JCAP* **08** (2007) 005 [[hep-th/0701199](#)].
- [16] T. Nishioka and T. Takayanagi, *Free Yang-Mills vs. Toric Sasaki-Einstein*, *Phys. Rev. D* **76** (2007) 044004 [[hep-th/0702194](#)].

- [17] M. Cadoni, *Entanglement entropy of two-dimensional Anti-de Sitter black holes*, *Phys. Lett. B* **653** (2007) 434 [[arXiv:0704.0140](#)].
- [18] V.E. Hubeny, M. Rangamani and T. Takayanagi, *A covariant holographic entanglement entropy proposal*, *JHEP* **07** (2007) 062 [[arXiv:0705.0016](#)].
- [19] J. Hammersley, *Numerical metric extraction in AdS/CFT*, *Gen. Rel. Grav.* **40** (2008) 1619 [[arXiv:0705.0159](#)].
- [20] S. Das, S. Shankaranarayanan and S. Sur, *Power-law corrections to entanglement entropy of black holes*, *Phys. Rev. D* **77** (2008) 064013 [[arXiv:0705.2070](#)].
- [21] G. Minton and V. Sahakian, *A new mechanism for non-locality from string theory: uV -IR quantum entanglement and its imprints on the CMB*, *Phys. Rev. D* **77** (2008) 026008 [[arXiv:0707.3786](#)].
- [22] M. Cadoni, *Induced gravity and entanglement entropy of 2D black holes*, [arXiv:0709.0163](#).
- [23] I.R. Klebanov, D. Kutasov and A. Murugan, *Entanglement as a probe of confinement*, *Nucl. Phys. B* **796** (2008) 274 [[arXiv:0709.2140](#)].
- [24] T. Azeyanagi, T. Nishioka and T. Takayanagi, *Near extremal black hole entropy as entanglement entropy via AdS2/CFT1*, *Phys. Rev. D* **77** (2008) 064005 [[arXiv:0710.2956](#)].
- [25] A. Faraggi, L.A. Pando Zayas and C.A. Terrero-Escalante, *Holographic entanglement entropy and phase transitions at finite temperature*, [arXiv:0710.5483](#).
- [26] D.V. Fursaev, *Entanglement entropy in quantum gravity and the plateau problem*, *Phys. Rev. D* **77** (2008) 124002 [[arXiv:0711.1221](#)].
- [27] V.E. Hubeny and M. Rangamani, *Holographic entanglement entropy for disconnected regions*, *JHEP* **03** (2008) 006 [[arXiv:0711.4118](#)].
- [28] H. Casini, *Entropy localization and extensivity in the semiclassical black hole evaporation*, [arXiv:0712.0403](#).
- [29] T. Azeyanagi, A. Karch, T. Takayanagi and E.G. Thompson, *Holographic calculation of boundary entropy*, *JHEP* **03** (2008) 054 [[arXiv:0712.1850](#)].
- [30] J.L.F. Barbon and C.A. Fuertes, *A note on the extensivity of the holographic entanglement entropy*, *JHEP* **05** (2008) 053 [[arXiv:0801.2153](#)].
- [31] T. Hirata, *New inequality for Wilson loops from AdS/CFT*, *JHEP* **03** (2008) 018 [[arXiv:0801.2863](#)].
- [32] A. Velytsky, *Entanglement entropy in $d+1$ SU(N) gauge theory*, *Phys. Rev. D* **77** (2008) 085021 [[arXiv:0801.4111](#)].
- [33] M.S. Berger and R.V. Buniy, *Entanglement entropy and spatial geometry*, *JHEP* **07** (2008) 095 [[arXiv:0801.4564](#)].
- [34] A. Schwimmer and S. Theisen, *Entanglement entropy, trace anomalies and holography*, *Nucl. Phys. B* **801** (2008) 1 [[arXiv:0802.1017](#)].
- [35] A.J.M. Medved, *A comment or two on holographic dark energy*, [arXiv:0802.1753](#).

- [36] S.N. Solodukhin, *Entanglement entropy, conformal invariance and extrinsic geometry*, *Phys. Lett. B* **665** (2008) 305 [[arXiv:0802.3117](#)].
- [37] P.V. Buividovich and M.I. Polikarpov, *Numerical study of entanglement entropy in SU(2) lattice gauge theory*, *Nucl. Phys. B* **802** (2008) 458 [[arXiv:0802.4247](#)].
- [38] J.L.F. Barbon and C.A. Fuertes, *Holographic entanglement entropy probes (non)locality*, *JHEP* **04** (2008) 096 [[arXiv:0803.1928](#)].
- [39] T. Hartman and A. Strominger, *Central charge for AdS₂ quantum gravity*, [arXiv:0803.3621](#).
- [40] A. Pakman and A. Parnachev, *Topological entanglement entropy and holography*, *JHEP* **07** (2008) 097 [[arXiv:0805.1891](#)].
- [41] G. Michalogiorgakis, *Entanglement entropy of two dimensional systems and holography*, [arXiv:0806.2661](#).
- [42] G. Vidal, J.I. Latorre, E. Rico and A. Kitaev, *Entanglement in quantum critical phenomena*, *Phys. Rev. Lett.* **90** (2003) 227902 [[quant-ph/0211074](#)].
- [43] A. Kitaev and J. Preskill, *Topological entanglement entropy*, *Phys. Rev. Lett.* **96** (2006) 110404 [[hep-th/0510092](#)].
- [44] M. Levin and X.G. Wen, *Detecting topological order in a ground state wave function*, *Phys. Rev. Lett.* **96** (2006) 110405 [[cond-mat/0510613](#)].
- [45] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, *The Hagedorn/deconfinement phase transition in weakly coupled large- N gauge theories*, *Adv. Theor. Math. Phys.* **8** (2004) 603 [[hep-th/0310285](#)];
B. Sundborg, *The Hagedorn transition, deconfinement and $N = 4$ SYM theory*, *Nucl. Phys. B* **573** (2000) 349 [[hep-th/9908001](#)].
- [46] A. Adams, J. Polchinski and E. Silverstein, *Don't panic! Closed string tachyons in ALE space-times*, *JHEP* **10** (2001) 029 [[hep-th/0108075](#)].
- [47] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [[hep-th/9803131](#)].
- [48] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)].
- [49] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)].
- [50] Y. Hikida, *Phase transitions of large- N orbifold gauge theories*, *JHEP* **12** (2006) 042 [[hep-th/0610119](#)].
- [51] D.N. Kabat, *Black hole entropy and entropy of entanglement*, *Nucl. Phys. B* **453** (1995) 281 [[hep-th/9503016](#)].
- [52] P. Basu and S.R. Wadia, *R -charged AdS₅ black holes and large- N unitary matrix models*, *Phys. Rev. D* **73** (2006) 045022 [[hep-th/0506203](#)].
- [53] D. Yamada and L.G. Yaffe, *Phase diagram of $N = 4$ super-Yang-Mills theory with R -symmetry chemical potentials*, *JHEP* **09** (2006) 027 [[hep-th/0602074](#)].

- [54] T. Harmark and M. Orselli, *Quantum mechanical sectors in thermal $N = 4$ super Yang-Mills on $R \times S^3$* , *Nucl. Phys. B* **757** (2006) 117 [[hep-th/0605234](#)].
- [55] K. Murata, T. Nishioka, N. Tanahashi and H. Yumisaki, *Phase transitions of charged Kerr-AdS black holes from large- N gauge theories*, [arXiv:0806.2314](#).
- [56] T. Harmark, K.R. Kristjansson and M. Orselli, *Decoupling limits of $N = 4$ super Yang-Mills on $R \times S^3$* , *JHEP* **09** (2007) 115 [[arXiv:0707.1621](#)].
- [57] S.S. Gubser, I.R. Klebanov and A.W. Peet, *Entropy and temperature of black 3-branes*, *Phys. Rev. D* **54** (1996) 3915 [[hep-th/9602135](#)].
- [58] S. Dong, E. Fradkin, R.G. Leigh and S. Nowling, *Topological entanglement entropy in Chern-Simons theories and quantum hall fluids*, *JHEP* **05** (2008) 016 [[arXiv:0802.3231](#)].
- [59] M.R. Douglas and V.A. Kazakov, *Large- N phase transition in continuum QCD in two-dimensions*, *Phys. Lett. B* **319** (1993) 219 [[hep-th/9305047](#)].
- [60] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, [hep-th/9603167](#).